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First-order linear equations

An equation of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

is a **linear equation** in y. The dependent variable y and its derivatives only occur to the first power, with coefficients which are functions of x alone.

Here is a **first-order linear** equation:

$$a(x)y' + b(x)y = c(x).$$

Divide through by $a(x)$:

$$y' + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}.$$

Rename the fractions:

$$y' + P(x)y = Q(x).$$

You should write first-order linear equations in this standard form before using the solution method below.

The idea for solving this equation is to try to turn the left side into an **exact** form — i.e. something which is exactly $\dfrac{df}{dx}$ for some f. To do this, multiply both sides by the **integrating factor**

$$I = \exp \int P(x)\,dx = e^{\int P(x)\,dx}.$$

Here is why it works. By the Product Rule and the Chain Rule,

$$\frac{d}{dx}\left(y \exp \int P(x)\,dx\right) = y'e^{\int P(x)\,dx} + y\frac{d}{dx}\left(e^{\int P(x)\,dx}\right) =$$

$$y'e^{\int P(x)\,dx} + ye^{\int P(x)\,dx} \cdot \frac{d}{dx}\left(\int P(x)\,dx\right) = y'e^{\int P(x)\,dx} + ye^{\int P(x)\,dx} \cdot P(x).$$

The last expression is just $e^{\int P(x)\,dx}$ times the left side of our original differential equation. So multiply the original equation by $e^{\int P(x)\,dx}$:

$$y'e^{\int P(x)\,dx} + yPe^{\int P(x)\,dx} = Qe^{\int P(x)\,dx} = IQ.$$

As above, the left side is the derivative of $ye^{\int P(x)\,dx}$, so

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) = IQ,$$

$$ye^{\int P(x)\,dx} = \int IQ\,dx,$$

$$yI = \int IQ\,dx.$$

In doing a problem, you can simply compute $I = e^{\int P(x)\,dx}$, then jump to the last equation. To finish, compute the integral on the right side.

Example. $\dfrac{dy}{dx} + \dfrac{3}{x}y = \dfrac{\cos x}{x^2}$.

First, compute the integrating factor:

$$I = \exp \int \frac{3}{x}\, dx = \exp 3 \ln x = \exp \ln x^3 = x^3.$$

(This cancellation of exp and ln often occurs in these computations. Note that you have to push the constant into the exponent first.)

Now plug the integrating factor into the equation $yI = \int IQ\, dx$:

$$yx^3 = \int x^3 \frac{\cos x}{x^2}\, dx = \int x \cos x\, dx.$$

Compute the integral on the right using integration by parts:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int dx \\ + & x & & \cos x \\ & & \searrow & \\ - & 1 & & \sin x \\ & & \searrow & \\ + & 0 & \rightarrow & -\cos x \end{array}$$

$$\int x \cos x\, dx = x \sin x + \cos x + C$$

Hence,

$$yx^3 = x \sin x + \cos x + C, \quad y = \frac{\sin x}{x^2} + \frac{\cos x}{x^3} + \frac{C}{x^3}. \quad \square$$

Example. $y' - \dfrac{\sin x}{\cos x}y = (\sin x)^5$, $y(0) = 1$.

The "$y(0) = 1$" is called an **initial condition**. This means you are to find the solution which satisfies $x = 0$, $y = 1$ — i.e. the solution which passes through the point $(0, 1)$. To do this, plug $x = 0$ and $y = 1$ into the general solution and solve for the arbitrary constant.

The integrating factor is

$$I = \exp\left(-\int \frac{\sin x}{\cos x}\, dx\right) = \exp \ln \cos x = \cos x.$$

Therefore,

$$y \cos x = \int (\sin x)^5 \cos x\, dx = \frac{1}{6}(\sin x)^6 + C,$$

$$y = \frac{1}{6}\frac{(\sin x)^6}{\cos x} + \frac{C}{\cos x}.$$

Now plug in the initial condition:

$$1 = \frac{1}{6} \cdot 0 + C, \quad \text{so} \quad C = 1.$$

The solution is

$$y = \frac{1}{6}\frac{(\sin x)^6}{\cos x} + \frac{1}{\cos x}. \quad \square$$

Example. $y\,dx + (3x - xy + 2)\,dy = 0$.

This equation is not linear in y:

$$\frac{dy}{dx} + \frac{y}{3x - xy + 2} = 0.$$

However, it is linear in x:

$$\frac{dx}{dy} + \frac{3}{y}x - x + \frac{2}{y} = 0, \quad \frac{dx}{dy} + \left(\frac{3}{y} - 1\right)x = -\frac{2}{y}.$$

The integrating factor is

$$I = \exp\int\left(\frac{3}{y} - 1\right)\,dy = y^3e^{-y}.$$

Therefore,

$$xy^3e^{-y} = -2\int y^2e^{-y}\,dy = -2\left(-y^2e^{-y} - 2ye^{-y} - 2e^{-y}\right) + C = 2y^2e^{-y} + 4ye^{-y} + 4e^{-y} + C.$$

The solution is

$$x = \frac{2}{y} + \frac{4}{y^2} + \frac{4}{y^3} + \frac{C}{y^3}e^y.$$

Here's the work for the integral:

	$\frac{d}{dy}$		$\int dy$
$+$	y^2		e^{-y}
		$\searrow$	
$-$	$2y$		$-e^{-y}$
		$\searrow$	
$+$	2		e^{-y}
		$\searrow$	e^{-y}
$-$	0	$\rightarrow$	$-e^{-y}$

$$\int y^2e^{-y}\,dy = -y^2e^{-y} - 2ye^{-y} - 2e^{-y} + C. \quad \square$$

Example. $y' = 2y + e^{2x}\cos 3x$, $y(0) = 4$.

Rewrite the equation as $y' - 2y = e^{2x}\cos 3x$.
The integrating factor is

$$I = \exp\int -2\,dx = e^{-2x}.$$

(*A standard mistake here is to use* 2 *instead of* -2. But the form I used in setting things up was $y' + P(x)y = Q(x)$, with a "+" on the left. So if the y term is subtracted, the "−" is used in computing I.)
Therefore,

$$ye^{-2x} = \int e^{-2x}e^{2x}\cos 3x\,dx = \int \cos 3x\,dx = \frac{1}{3}\sin 3x + C.$$

The general solution is

$$y = \frac{1}{3}e^{2x}\sin 3x + Ce^{2x}.$$

Plug in the initial condition:

$$4 = y(0) = 0 + C, \quad C = 4.$$

The solution is

$$y = \frac{1}{3}e^{2x}\sin 3x + 4e^{2x}. \quad \Box$$

Example. **(Discontinuous forcing)** $y' + \frac{3}{x}y = g(x)$, where

$$g(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}, \quad \text{and} \quad y\left(\frac{1}{2}\right) = \frac{1}{8}.$$

The idea is to solve the equation separately on $0 \le x \le 1$ and on $x > 1$, then match the pieces up at $x = 1$ to get a *continuous* solution.

$0 \le x \le 1$: $y' + \frac{3}{x}y = 1$. The integrating factor is

$$I = \exp\int \frac{3}{x}\,dx = e^{3\ln x} = x^3.$$

Then

$$yx^3 = \int x^3\,dx = \frac{1}{4}x^4 + C.$$

The solution is

$$y = \frac{1}{4}x + \frac{C}{x^3}.$$

Plug in the initial condition:

$$\frac{1}{8} = y\left(\frac{1}{2}\right) = \frac{1}{8} + 8C, \quad C = 0.$$

The solution on the interval $0 \le x \le 1$ is

$$y = \frac{1}{4}x.$$

Note that $y(1) = \frac{1}{4}$.

$x > 1$: $y' + \frac{3}{x}y = \frac{1}{x}$. The integrating factor is the same as before, so

$$yx^3 = \int x^2\,dx = \frac{1}{3}x^3 + C.$$

The solution is

$$y = \frac{1}{3} + \frac{C}{x^3}.$$

In order to get this piece to "match" with the previous piece, set $y(1) = \frac{1}{4}$:

$$\frac{1}{4} = y(1) = \frac{1}{3} + C, \quad C = -\frac{1}{12}.$$

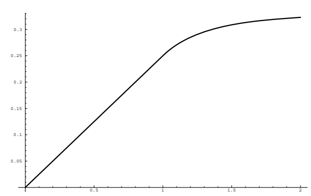
The solution on the interval $x > 1$ is

$$y = \frac{1}{3} - \frac{1}{12}\frac{1}{x^3}.$$

The complete solution is

$$y = \begin{cases} \frac{1}{4}x & \text{if } 0 \le x \le 1 \\ \frac{1}{3} - \frac{1}{12}\frac{1}{x^3} & \text{if } x > 1 \end{cases}$$

You can see the two pieces glued together in the picture below:



$\square$

Example. Calvin Butterball's backpack has a capacity of 5 gallons. Calvin's creative lab partners pour 1 gallon of pure water into the backpack. After that, water containing 0.5 pounds of dissolved salt per gallon is pumped in at 2 gallons per minute; the well-stirred mixture drains out the bottom at 1 gallon per minute. How many pounds of salt are dissolved in the solution in the backpack at the instant when it overflows?

Let S be the amount of salt (in pounds) dissolved in the solution in the backpack at time t. Write down the **rate equation** for S; it is the inflow rate minus the outlow rate:

$$\frac{dS}{dt} = \left(0.5\frac{\text{lbs}}{\text{gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right) - \left(\frac{S\text{ lbs}}{1+t\text{ gal}}\right)\left(1\frac{\text{gal}}{\text{min}}\right).$$

In both terms, I've multiplied the concentration by the flow rate. Everything is straightforward except perhaps the $\frac{S}{1+t}$ term. This is the concentration of salt in the tank at time t. Why? First, S *is* the amount of salt (in pounds) dissolved in the solution. Now there is 1 gallon in the backpack initially, and the volume increases by $2 - 1 = 1$ gallon each minute — so after t minutes, there are $1 + t$ gallons. Thus, the concentration is $\frac{S}{1+t}$. This goes into the outflow term, because it's the concentration of salt in the fluid draining out.

Notice that $\frac{dS}{dt}$ has the units pounds per minute. And if you cancel the gallon units on the right side, everything on the right has the units pounds per minute as well. This serves as a check that you've written down something sensible.

Rearrange the equation:

$$\frac{dS}{dt} = 1 - \frac{S}{1+t}, \quad \frac{dS}{dt} + \frac{S}{1+t} = 1.$$

Find the integrating factor:

$$I = \exp\int \frac{1}{1+t}\,dt = 1 + t.$$

Therefore,

$$S(1+t) = \int (1+t)\,dt = \frac{1}{2}(1+t)^2 + C.$$

When $t = 0$, $S = 0$ (because there was pure water in the backpack initially):

$$0 \cdot 1 = \frac{1}{2} + C, \quad C = -\frac{1}{2}.$$

Now

$$S(1+t) = \frac{1}{2}(1+t)^2 - \frac{1}{2},$$

$$S = \frac{1}{2}(1+t) - \frac{1}{2}\frac{1}{1+t}.$$

Finally, when does the backpack overlow? The capacity is 5 gallons, there is 1 gallon initially, and the volume increases by 1 gallon each minute. Hence, it overlows when $t = 4$:

$$S = \frac{1}{2}(1+4) - \frac{1}{2}\frac{1}{1+4} = 2.4 \text{ pounds.} \quad \Box$$

It is important to know when a differential equation has a solution (the **existence problem**). Some equations have solutions only for certain sets of initial conditions; I'll give an example below of an equation with *no solutions*.

It's also important to know, if a solution is found, whether it is the only possible solution. Geometrically, the question is whether there is a single solution curve passing through a given point. This is called the **uniqueness problem**. I will give an example later on of an equation with infinitely many solution curves passing through a point.

These questions can be answered for certain classes of differential equations.

Consider the initial value problem

$$y' + P(x)y = Q(x), \qquad y(x_0) = y_0.$$

The **existence and uniqueness theorem** for first-order linear equations says that if P and Q are continuous on an interval (a, b), then there is a *unique* solution satisfying the initial condition.

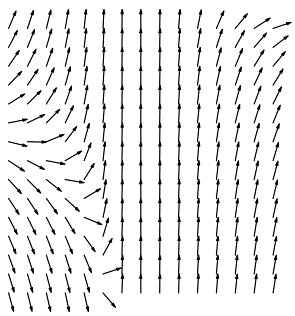
Example. $\dfrac{dy}{dx} + \dfrac{3}{x-1}y = 6\dfrac{x+1}{(x-1)^2}$, $y(2) = 9$.

$P = \dfrac{3}{x-1}$; it is continuous for $x > 1$ and for $x < 1$. $Q = 6\dfrac{x+1}{(x-1)^2}$; it is continuous for $x > 1$ and for $x < 1$. The initial condition is $x = 2$, $y = 9$. Since $x = 2$ lies in the interval $x > 1$, there is a unique solution to the initial value problem for $x > 1$. *Notice that you know this without solving the equation!*

In fact, the solution is

$$y = \frac{2x^3 - 6x + 5}{(x-1)^3}, \qquad x > 1.$$

Here is the direction field for the equation:



Notice the singularity along the line $x = 1$. □

Example. $xy' + 2y = 3x$ has only one solution defined at $x = 0$.

To see this, rewrite the equation as $y' + \frac{2}{x}y = 3$.

The integrating factor is

$$I = \exp \int \frac{2}{x}\,dx = x^2.$$

Then

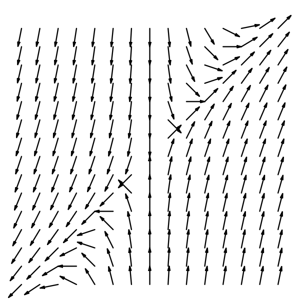
$$yx^2 = \int 3x^2\,dx = x^3 + C.$$

The solution is

$$y = x + \frac{C}{x^2}.$$

$y = x$ is a solution (set $C = 0$), and it's defined at $x = 0$. However, if $C \neq 0$, the solution is not defined at $x = 0$.

Here's the direction field:



Notice that the solution $y = x$ is the only solution that crosses the singularity at $x = 0$.

On the other hand, the solutions change dramatically if the equation is changed just a little. Consider $xy' - 2y = 3x$. Rewrite it as $y' - \frac{2}{x}y = 3$.

The integrating factor is

$$I = \exp \int -\frac{2}{x}\,dx = x^{-2}.$$

Then

$$yx^{-2} = \int \frac{3}{x^2}\,dx = -\frac{3}{x} + C.$$

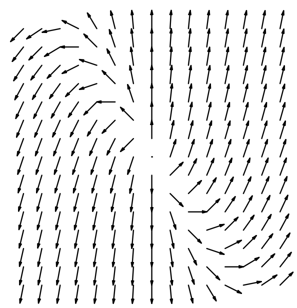
The solution is

$$y = -3x + Cx^2.$$

In this case, y is defined for all x, for all values of C.

However, note that the initial value problem $y(0) = 0$ has infinitely many solutions, since $y(0) = 0$ for any value of C. On the other hand, if $y_0 \neq 0$, the initial value problem $y(0) = y_0$ has no solutions.

Here's the direction field:



You can see all the solutions curves emanating from the origin, corresponding to the infinitely many solutions to the initial value problem with $y(0) = 0$. $\square$

The existence and uniqueness theorem stated above applies to first-order linear equations. There are similar results for other classes of equations. However, the situation for an arbitrary differential equation is often not so nice.

Example. (An equation with exactly one solution) Suppose that

$$\left(\frac{dy}{dx}\right)^2 + y^2 = 0 \quad \text{for} \quad a < x < b.$$

A sum of squares is 0 if and only if both terms are 0. Therefore, $y = 0$, and this is the only possible solution. $\square$

Example. (An equation with no solutions) Suppose that

$$\left(\frac{dy}{dx}\right)^2 + x^2 = 0$$

has a solution $y = f(x)$ on the interval $a < x < b$. Then $x = 0$ for $a < x < b$, which is ridiculous. $\Rightarrow\Leftarrow$ Hence, the equation has no solution on any open interval. $\square$

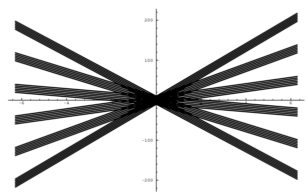
Example. $\cos y' = 0$.

$\cos \text{junk} = 0$ when junk is an odd multiple of $\frac{\pi}{2}$. Thus,

$$y' = (2n+1)\frac{\pi}{2}, \quad \text{so} \quad y = (2n+1)\frac{\pi}{2}x + C, \qquad n \in \mathbf{Z}.$$

This is actually a two-parameter family of solutions. That is $y = \dfrac{5\pi}{2}x + C$ is a family of solutions, $y = \dfrac{\pi}{2}x + C$ is a family of solutions, and so on. In this situation, there are infinitely many solutions passing through each point.

I've drawn the family of solutions curves below for n and c each going from -10 to 10 in increments of 4.



You can see that it looks as though many curves pass through a given point. $\square$

